# Solid Möbius strips as algebraic surfaces

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**Abstract**

We give explicit polynomials in three real variables $x$, $y$ and $z$ such that the zero sets have the shape of solid Möbius strips. The polynomials depend on a further parameter which enables a deformation from an embedded torus. We use only elementary methods such that the proofs are also accessible to graduate math work groups for pupils in secondary schools. The results can be easily visualized using the free SURFER software of Oberwolfach.

**Keywords:** Möbius strip, torus, algebraic surface, rotation, double angle, triple angle, de Moivre's formula
**MSC:** 14J25, 14Q10, 51N10, 57N05, 57N35

## 1 Introduction

The Möbius strip [1] is a well-known classical object in geometry and topology which attracts not only professional mathematicians, but many people. It can be obtained by glueing a strip of paper with a twist (i.e., a rotation by $\pi$) of two parallel edges of the strip to be identified. If one rotates the edges before glueing by $k\pi$ with $k$ an odd integer, one obtains different embeddings $M_k$ of the Möbius strip in 3-space $\mathbb{R}^3$. If $k$ is even, the resulting surface is a twisted cylinder surface. We call $M_k$ a $k$-twisted Möbius strip.

As $M_k$ is a non-orientable compact surface with boundary, it cannot be represented as the inverse set $f^{-1}(a)$ of a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ (or more genereral $(f, g) : \mathbb{R}^3 \to \mathbb{R} \times [0, 1]$) and a regular value $a \in \mathbb{R}$ of $f$ [2].

Instead we will explicitly construct polynomials $p(x, y, z)$ in three real variables $x$, $y$ and $z$ such that the zero set $p^{-1}(0)$ has the shape of a **solid Möbius strip**. Here, a solid Möbius strip $N_k$ is defined as the boundary of a small smooth tubular neighborhood of $M_k$. As $N_k$ is a closed surface embedded in $\mathbb{R}^3$, $N_k$ is framed and orientable by the generalized Jordan separation theorem [3], and it can be represented as the inverse set of a smooth function with a regular value by the Pontrjagin-Thom Construction [4]. By the Weierstrass approximation theorem [5], there exists also a polynomial function representing a surface with the shape of $N_k$. But these theorems prove only existance and do not help to construct explicit polynomials.

It is the purpose of this paper to construct such explicit polynomials by elementary methods. Moreover, our polynomials will also depend on a real parameter $a$ such that $a = 1$ gives the standard torus embedding in $\mathbb{R}^3$ and $a \mapsto 0$ describes a deformation of the torus to $N_k$. We will achieve this using a thin ellipse as cross-section because it looks similar as a transversal line segment within the Möbius strip.

In the last section, we give some hints how to visualize the surfaces by the free SURFER software of Oberwolfach which also allows real-time deformation by changing some surface parameters. The pictures of this article are all created with this software.

It is a pleasure for me to thank Gert-Martin Greuel and Andreas Matt for introducing me to the SURFER and its fantastic visualization features which I could explore with a graduate math work group for pupils ("Mathe-AG") in 2008, as a cooperation of the Mathematisches Forschungsinstitut Oberwolfach (MFO) with the nearby secondary schools, Robert-Gerwig-Gymnasium (Hausach) and Technisches Gymnasium (Wolfach). Moreover, I like to thank also Oliver Labs and Thomas Markwig for organizing with me an advanced training for schoolteachers on the subject of visualization of algebraic curves and surfaces at the MFO in November 2008. The author's construction of the 1- and 2-twisted Möbius strip were presented at this training.

## 2 The $k$-twisted solid Möbius strip

We denote by $d$ the distance of a point $(x, y) \in \mathbb{R}^2$ to the origin and by $\phi \in [0, 2\pi[$ its angle to the $x$-axis, i.e.

$$d^2 = x^2 + y^2, \quad x = d\cos(\phi) \quad \text{and} \quad y = d\sin(\phi).$$

We denote $C := \cos(\phi)$ and $S := \sin(\phi)$.

Now we consider a second coordinate system $(t, z) \in \mathbb{R}^2$ and an ellipse $E$ with center $(1, 0)$ and length of the main radii given by $\sqrt{b}$ and $\sqrt{a}$, i.e. $E$ is given by the equation

$$\frac{(t-1)^2}{b} + \frac{z^2}{a} = 1$$

where we assume $a$ and $b$ to be positive real numbers. Multiplying this with $ab$ gives

$$a(t-1)^2 + bz^2 = ab.$$

If we keep the center $(1,0)$ but rotate the ellipse by an angle $\psi \in \mathbb{R}$, this can be achieved by a coordinate rotation

$$(t-1) \mapsto c(t-1) + sz \qquad z \mapsto -s(t-1) + cz,$$

where $c := \cos(\psi)$ and $s := \sin(\psi)$. The equation of the rotated ellipse $E_\psi$ is then given by

$$a(c(t-1) + sz)^2 + b(-s(t-1) + cz)^2 = ab,$$

which is equivalent to

$$c^2(a(t-1)^2 + bz^2) + 2cs(a-b)(t-1)z + s^2(b(t-1)^2 + az^2) = ab.$$

Now we consider the $(t,z)$ coordinate system as rotating around the $z$-axis, while the rotating $t$-axis spans the $(x,y)$-plane. At the same time the $t$-axis rotates by an angle $\phi$ in the $(x,y)$-plane, we let rotate the ellipse $E_\psi$ around $(1,0)$ in the $(t,z)$-plane by the angle

$$\psi = \frac{k}{2}\phi.$$

This gives exactly the behaviour of $k$-times twisting a solid strip by the angle $\pi$ before glueing together the ends of the solid strip (which we are assuming to have the shape of an ellipse). Then a semi-explicite parametrization of a solid Möbius strip $N_k$ is defined by the equation above for $E_\psi$ together with

$$t^2 = x^2 + y^2, \quad x = t\cos(\phi) \quad \text{and} \quad y = t\sin(\phi)$$

and $\psi = \frac{k}{2}\phi$.

We note that we will obtain a closed smooth surface without self-intersection if $a$ and $b$ are smaller than 1. Otherwise the ellipse $E_\psi$ can reach some negative coordinate values $t$ (for suitable $\psi$) which can produce self-intersections because of the rotation around the $z$-axes.

As a special case, we obtain the **standard torus** for $a = b < 1$, and in this case $k$ clearly does not play any role. Then the equations simplify to

$$t^2 = x^2 + y^2 \quad \text{and} \quad (t-1)^2 + z^2 = a.$$

## 3 Solid Möbius strips by polynomial equations

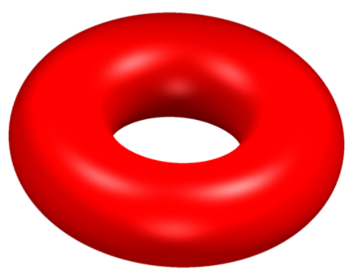
We will construct an implicit polynomial representation $p(x, y, z)$ by elimination of the variables $\phi$ (i.e., $C$ and $S$), $\psi$ (i.e., $c$ and $s$) and $t$.

The elimination is very easy in the case of the standard torus as there is no twisting. From the equation $(t-1)^2 + z^2 = a$ we get

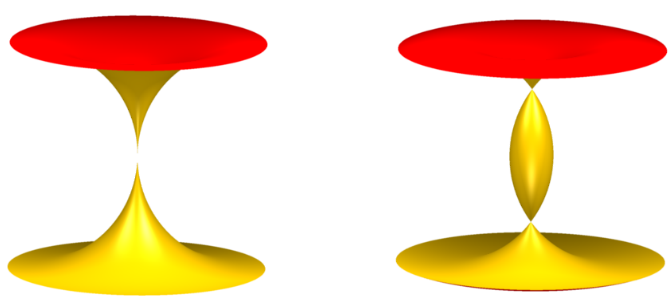
$$t^2 + z^2 + 1 - a = 2t.$$

Squaring this equation and insertion of $t^2 = x^2 + y^2$ yields the **torus polynomial equation** of degree 4:

$$(x^2 + y^2 + z^2 + 1 - a)^2 = 4(x^2 + y^2).$$



For $a \geq 1$ there are interesting self-intersections on the $z$-axes which can also be visualized by the SURFER software (see the last section).



The next easiest example is not given by the 'classical' (i.e., 1-twisted) solid Möbius strip $N_1$, but by the 2-twisted solid Möbius strip $N_2$. In this case, we have $\phi = \psi$, showing that $x = ct$ and $y = st$. Now we use the equation for $E_\psi$, insert $c = \frac{x}{t}$ and $s = \frac{y}{t}$, and multiply the equation with $t^2$ in order to clear the denominators:

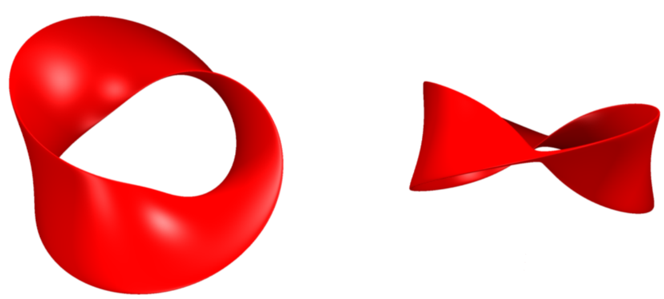
$$x^2(a(t-1)^2 + bz^2) + 2xy(a-b)(t-1)z + y^2(b(t-1)^2 + az^2) = abt^2.$$

As in the case of the standard torus, it is easy to get now an implicit polynomial equation: First collect the terms with even powers of $t$ on the left side and the terms with the odd powers on the right side

$$(t^2+1)(ax^2+by^2) + z^2(bx^2+ay^2) - 2(a-b)xyz - abt^2$$
$$= 2t(ax^2+by^2 - xyz(a-b)),$$

then square this equation and insert $t^2 = x^2+y^2$. This gives the **polynomial equation** $(N_2)$ **for the** 2**-twisted solid Möbius strip** of degree 8:

$$\big((x^2+y^2+1)((ax^2+by^2)+z^2(bx^2+ay^2)-2(a+b)xyz-ab(x^2+y^2)\big)^2$$
$$= 4(x^2+y^2)(ax^2+by^2-xyz(a+b))^2.$$



In order to obtain an analogeous polynomial equation for $N_1$, we have first to translate the relation $\psi = \frac{1}{2}\phi$ to the trigonometrical coefficients $c = \cos(\psi)$, $s = \sin(\psi)$ and $C = \cos(\phi)$, $S = \sin(\phi)$ by the formulas for the double angle:

$$C = c^2 - s^2 \quad \text{and} \quad S = 2cs.$$

Because of $c^2 + s^2 = 1$ we obtain $c^2 = \frac{1}{2}(1+C)$ and $s^2 = \frac{1}{2}(1-C)$. We insert this with $C = \frac{x}{t}$ and $S = \frac{y}{t}$ into the equation of $E_\psi$ and multiply with $2t$ in order to clear denominators:

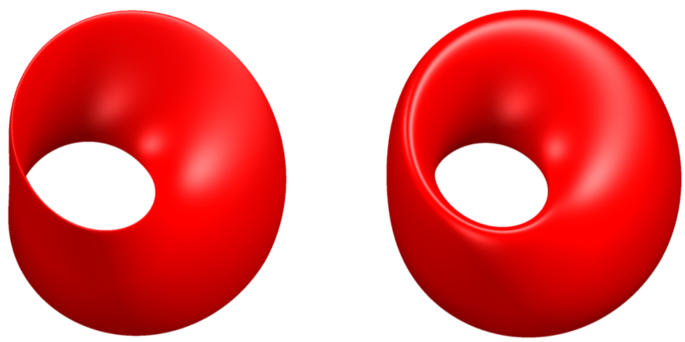
$$(t+x)(a(t-1)^2+bz^2) + 2y(a-b)(t-1)z + (t-x)(b(t-1)^2+az^2) = ab.$$

Again we collect the terms with even powers of $t$ on the left side and odd powers on the right side, where we factor out $a \pm b$ and $t$ from the odd powers:

$$(a-b)(x(t^2-z^2+1)-2yz)-(2a+2b+ab)t^2$$
$$=-t\big((a+b)(t^2+z^2+1)+2(a-b)(yz-x)\big).$$

Then we square this equation and insert $t^2 = x^2+y^2$, which yields the **polynomial equation** $(N_1)$ **for the 'classical' solid Möbius strip** of degree 6:

$$\big((a-b)(x(x^2+y^2-z^2+1)-2yz)-(2a+2b+ab)(x^2+y^2)\big)^2$$
$$=(x^2+y^2)\big((a+b)(x^2+y^2+z^2+1)+2(a-b)(yz-x)\big)^2.$$



As a last example for Möbius strips, we consider the equation of the 3-twisted solid Möbius strip $(N_3)$ which for example appears in the logo of the Mathematisches Forschungsinstitut Oberwolfach (see the web site www.mfo.de):

Here, the relation $\psi = \frac{3}{2}\phi$ gives with the formulas for the double angle and for the triple angle the following relations:

$$C^3-3CS^2=c^2-s^2 \quad \text{and} \quad 3C^2S-S^3=2cs.$$

Because of $c^2 + s^2 = 1$ we obtain $c^2 = \frac{1}{2}(1 + C^3 - 3CS^2)$ and $s^2 = \frac{1}{2}(1 - C^3 + 3CS^2)$, hence

$$c^2 = \frac{t^3 + x^3 - 3xy^2}{2t^3}, \quad s^2 = \frac{t^3 - x^3 + 3xy^2}{2t^3} \quad \text{and} \quad cs = \frac{3x^2y - y^3}{2t^3}.$$

Inserting this into equation $(E_\psi)$ and multiplying with $2t^3$ in order to clear denominators gives

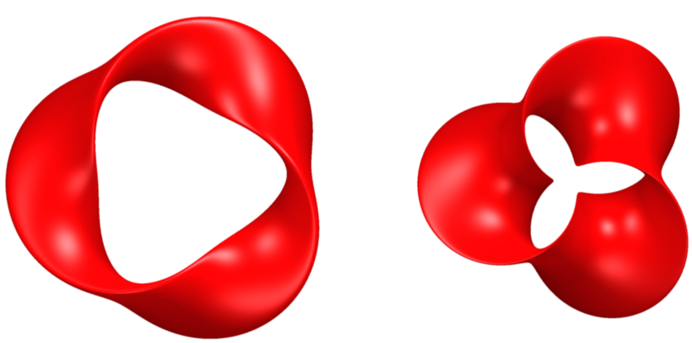
$$(t^3 + x^3 - 3xy^2)(a(t-1)^2 + bz^2) + 2(3x^2y - y^3)(a-b)(t-1)z$$
$$+(t^3 - x^3 + 3xy^2)(b(t-1)^2 + az^2) = 2abt^3.$$

Separating even and odd powers yields

$$-2(a+b)t^4 + (a-b)((x^3 - 3xy^2)(t^2 + 1 - z^2) - 2(3x^2y - y^3)z)$$
$$= -t\big((a+b)t^2(t^2 + 1 + z^2) - 2(a-b)(x^3 - 3xy^2 - z(3x^2y - y^3)) - 2abt^2\big)$$

and squaring and inserting $t^2 = x^2 + y^2$ yields the **polynomial equation** $(N_1)$ **for the 3-twisted solid Möbius strip** of degree 10:

$$\big(-2(a+b)(x^2+y^2)^2 + (a-b)((x^3-3xy^2)(x^2+y^2+1-z^2) - 2(3x^2y-y^3)z)\big)^2$$
$$= (x^2+y^2)\big((a+b)(x^2+y^2)(x^2+y^2+1+z^2) -$$
$$-2(a-b)(x^3 - 3xy^2 - z(3x^2y - y^3)) - 2ab(x^2+y^2)\big)^2.$$



It is clear how to proceed for the $k$-twisted solid Möbius strips for $k \geq 4$. Here we have to use the de Moivre's formula [6] for the $k$-fold angle

$$\cos(k\phi) = C^k - \binom{k}{2}C^{k-2}S^2 + \binom{k}{4}C^{k-4}S^4 \mp \ldots =: a_k(C,S),$$

$$\sin(k\phi) = kC^{k-1}S - \binom{k}{3}C^{k-3}S^3 + \binom{k}{5}C^{k-5}S^5 \mp \ldots =: b_k(C,S).$$

If $k$ is even, we have $\psi = l\phi$ with $l := \frac{k}{2}$ and this leads to

$$c = \frac{a_l(x,y)}{t^l} \qquad s = \frac{b_l(x,y)}{t^l}.$$

If $k$ is odd, we have $2\psi = k\phi$ which yields

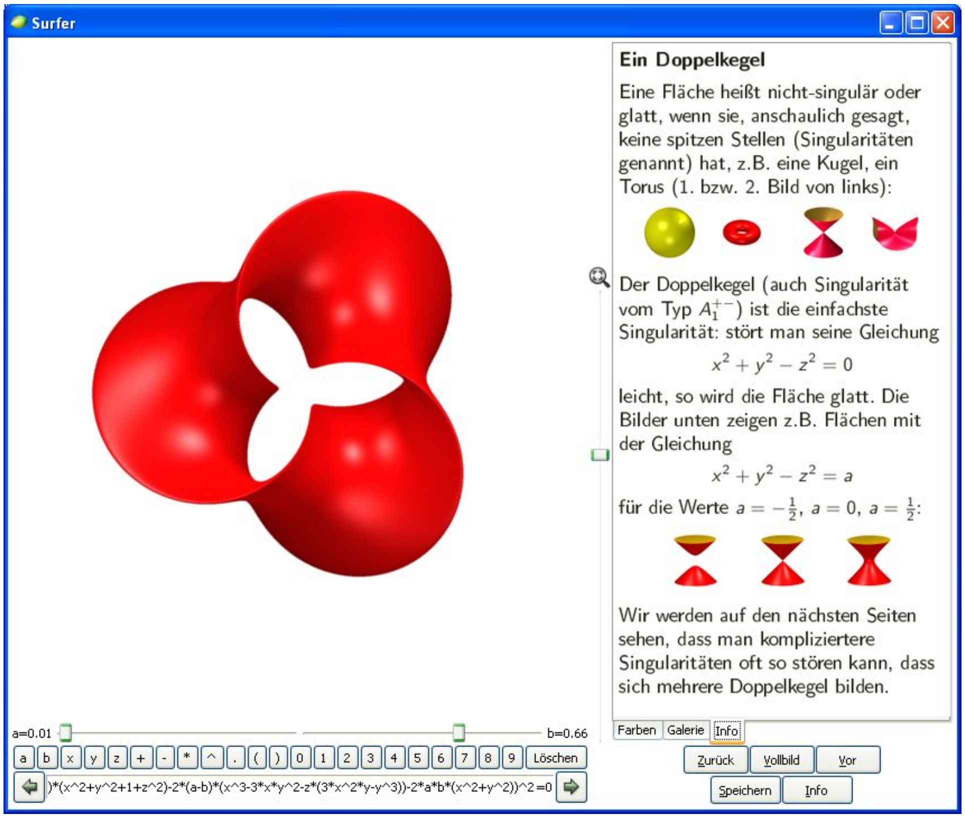
$$c^2 = \frac{t^k + a_k(x,y)}{2t^k}, \quad s^2 = \frac{t^k - a_k(x,y)}{2t^k} \quad \text{and} \quad cs = \frac{b_k(x,y)}{2t^k}$$

using similar arguments as before. Then we have to insert these expressions into the ellipse equation, separate even and odd powers of $t$, square the equation and insert $t^2 = x^2 + y^2$ in order to eliminate $t$. This gives a polynomial equation of order $2(k+2)$ for the $k$-twisted solid Möbius strip.

## 4 Visualization by the SURFER software

All pictures were generated with the free SURFER software which can be downloaded freely from the web site of the Mathematisches Forschungsinstitut Oberwolfach (see www.mfo.de and follow the link to the IMAGINARY web site or directly http://www.imaginary2008.de/surfer.php).

Installation and working with the SURFER is straight forward. One can just insert a polynomial in the three variables $x$, $y$ and $z$ and gets a real time visualisation of the zero set, which in general is a surface with some singularities and self-intersections. Using the cursor and a scroll bar, the surface can be rotated in every direction and the visible sector can be rescaled. It is also possible to coulor the surface with a large scale of available coulors. Moreover, one can use two parameters $a$ and $b$ in the polynomial which are fixed by two scroll bars and which allow real-time deformations of the zero set surface. Short tutorials are available on the SURFER web site, too. Because of its easy and intuitive use, SURFER can be strongly recommended to graduate math work groups for pupils in secondary schools.




In our case, the formulas contain the two variables $a$, $b$ which control the axes of the rotating ellipse. The scroll bar allows the fixing of values from 0 to 1. The parameter values $a = 0$ or $b = 0$ produce singular results and lead to visualisation errors with the SURFER. Therefore it makes sense to add a small offset to the parameter $a$, e.g. replace $a$ by $0.01 + a$. If one is interested in self-intersections, one can also use larger parameters by $2a$.

For the convenience of the reader, we have collected here some of the formulas above in a format which is ready for input to the SURFER.

**Torus** (we have multiplied the parameter $a$ by 2 in order to have self-intersections also available):

$$(x\wedge 2 + y\wedge 2 + z\wedge 2 + 1 - 2 * a)\wedge 2 - 4 * (x\wedge 2 + y\wedge 2)$$

**1-twisted Möbiusstrip:**

$$((a-b)*(x*(x\wedge 2+y\wedge 2-z\wedge 2+1)-2*y*z)-(2*a+2*b+a*b)*(x\wedge 2+y\wedge 2))\wedge 2$$

$$-(x\wedge 2+y\wedge 2)*((a+b)*(x\wedge 2+y\wedge 2+z\wedge 2+1)+2*(a-b)*(y*z-x))\wedge 2$$

**2-twisted Möbiusstrip:**

$$((x\wedge 2 + y\wedge 2 + 1) * (a * x\wedge 2 + b * y\wedge 2) + z\wedge 2 * (b * x\wedge 2 + a * y\wedge 2)$$
$$-2 * (a + b) * x * y * z - a * b * (x\wedge 2 + y\wedge 2))\wedge 2$$
$$-4 * (x\wedge 2 + y\wedge 2) * (a * x\wedge 2 + b * y\wedge 2 - x * y * z * (a + b))\wedge 2$$

**3-twisted Möbiusstrip:**

$$(-2*(a+b)*(x\wedge 2+y\wedge 2)\wedge 2+(a-b)*((x\wedge 3-3*x*y\wedge 2)*(x\wedge 2+y\wedge 2+1-z\wedge 2)$$
$$-2 * (3 * x\wedge 2 * y - y\wedge 3) * z))\wedge 2$$
$$-(x\wedge 2 + y\wedge 2) * ((a + b) * (x\wedge 2 + y\wedge 2) * (x\wedge 2 + y\wedge 2 + 1 + z\wedge 2)$$
$$-2*(a-b)*(x\wedge 3-3*x*y\wedge 2-z*(3*x\wedge 2*y-y\wedge 3))-2*a*b*(x\wedge 2+y\wedge 2))\wedge 2$$

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## References

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For pupils: see also the Wikipedia article for the Weierstrass Approximation Theorem.

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For pupils: see also the Wikipedia article for the de Moivre's formula.